SCALING FUNCTION FOR THE CRITICAL SPECIFIC HEAT IN A CONFINED GEOMETRY : SPHERICAL LIMIT

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Abstract

The scaling function for the critical specific heat is obtained exactly for temperatures above the bulk transition temperature by working in the spherical limit. Generalization of the function to arbitrary α (the specific heat exponent), gives an excellent account of the experimental data of Mehta and Gasparini near the superfluid transition.

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One of the primary theoretical tools for a strong coupling problem is the spherical limit and the $\frac{1}{n}$ expansion [1-5]. In critical phenomena, it has often provided useful insight in the past and is still one of the first techniques to be tried when a new kind of critical phenomenon surfaces [6-11]. A problem that has been receiving a fair amount of experimental and theoretical attention is that of the specific heat of liquid helium near the lambda point in a confined geometry. This is a very effective way of studying the finite size effects (FSE) since by going close enough to the lambda point (particularly easy in a space-shuttle experiment), the correlation length can be made larger than the confining length if the liquid is enclosed in a thin wafer geometry. The bulk specific heat C_p diverges near the lambda point as $\xi^{\alpha/\nu}$, where ξ is the correlation length proportional to $\mid T - T_c \mid^{-\nu}$. For the finite size system, the specific heat remains finite at $T = T_c$, with the value proportional to $L^{\alpha/\nu}$ where L is the confining length. For a given L, the specific heat changes from a $\xi^{\alpha/\nu}$ to a $L^{\alpha/\nu}$ as one approaches the bulk lambda point. The change from one regime to the other is described in terms of a scaling function $f(x)$ in terms of which we can write the specific heat as

$$C_p(\xi, L) = C_0\xi^{\alpha/\nu} f(\xi/L) + Constant \tag{1}$$

where the function $f(x)$ has the property that $f(0) = 1$ and $f(x) \sim x^{-\alpha/\nu}$ for $x \gg 1$. It is the function $f(x)$ that has been the object of several experimental and numerical investigations. In this letter, we determine $f(x)$ in the spherical limit. We also show how the result can be put to practical use.

We consider a n-component Ginzburg-Landau model with the free energy

functional

$$F = \int d^D r \ [\frac{m^2}{2} \sum_{i=1}^{n} \phi_i^2 + \frac{1}{2} \sum_{i=1}^{n} (\nabla \phi_i)^2 + \lambda \, (\sum_{i=1}^{n} \phi_i^2)^2 \,] \tag{2}$$

where $m^2 \sim (T - T_0)$, T_0 being a transition temperature and the coupling constant λ is $O(n^{-1})$ so that the quartic term remains the same order as the quadratic term when $n \to \infty$ (the spherical limit). We consider the system confined in one direction (we will call this the z - direction) within an extension L and the boundary conditions will be taken to be of Dirichlet type , i.e., $\phi_i = 0$ at $z = 0$ and $z = L$. The expansion in Fourier modes for $\phi_i(\vec{r})$ is

$$\phi_i(\vec{r}, z) = \frac{1}{\sqrt{L}} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \ \sum_n \phi_{in}(\vec{k}) \exp i\vec{k}.\vec{R} \, sin \, \frac{n\pi z}{L} \tag{3}$$

The specific heat is given by the correlation function

$$\begin{aligned} C_p &= \frac{1}{AL} \int d^{D-1}r_1 d^{D-1}r_2 \, dz_1 \, dz_2 < \sum_i \phi_i^2(\vec{r_1}, z_1) \sum_j \phi_j^2(\vec{r_2}, z_2) > \\ &= \frac{1}{4L} \sum_{n_1, n_2} \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \ \frac{d^{D-1}p_2}{(2\pi)^{D-1}} < \sum_i \phi_{in_1}(\vec{p_1}) \, \phi_{in_1}(-\vec{p}_1) \\ & \quad \sum_j \phi_{jn_2}(\vec{p_2}) \, \phi_{jn_2}(-\vec{p}_2) > \end{aligned} \tag{4}$$

The free energy of Eq(2) written in terms of the fields ϕ_{in} because

$$\begin{aligned} F &= \sum_n \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \, [\, \frac{1}{2}(m^2 + k^2 + \frac{n^2\pi^2}{L^2}) \, \phi_{in}(k) \, \phi_{in}(-k) \,] \\ & \quad + \frac{\lambda}{4} \sum_i \sum_j \int \frac{d^{D-1}k_1}{(2\pi)^{D-1}} \ \frac{d^{D-1}k_2}{(2\pi)^{D-1}} \ \frac{d^{D-1}k_3}{(2\pi)^{D-1}} \ dz \\ & \quad \phi_i(k_1, z) \, \phi_i(k_2, z) \, \phi_j(k_3, z) \, \phi_j(-k_1 - k_2 - k_3, z) \end{aligned} \tag{5}$$

and the averaging shown in Eq(4) has to be done with this free energy functional.

The Gaussian limit has to be disposed of first i.e. the limit where $\lambda = 0$ and the action is quadratic.

$$\begin{aligned}
C_p^{Gaussian} &= \frac{1}{4L} \sum_{n_1,n_2} \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \frac{d^{D-1}p_2}{(2\pi)^{D-1}} < \phi_{i,n_1}(\vec{p_1})\, \phi_{j,n_2}(\vec{p_2}) > \\
&\quad < \phi_{i,n_1}(\vec{p_1})\, \phi_{j,n_2}(\vec{p_2}) > \\
&= \frac{N}{2L} \sum_n \int \frac{d^{D-1}p}{(2\pi)^{D-1}} [< \phi_{i,n}(\vec{p})\, \phi_{i,n}(-\vec{p})]^2 \\
&= \frac{N}{2L} \sum_n \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{(p^2 + m^2 + \frac{n^2\pi^2}{L^2})} \\
&= \frac{NL^4}{2L} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} [\frac{coth\sqrt{p^2L^2 + m^2L^2}}{2(p^2L^2 + m^2L^2)^{3/2}} \\
&\quad - \frac{1}{(p^2L^2 + m^2L^2)^2} + \frac{cosech^2\sqrt{p^2L^2 + m^2L^2}}{2(p^2L^2 + m^2L^2)}] \\
&= \frac{N}{2} I_D(mL)
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
I_D(mL) &= L^3 \int \frac{d^{D-1}p}{(2\pi)^{D-1}} [\frac{coth\sqrt{p^2L^2 + m^2L^2}}{2(p^2L^2 + m^2L^2)^{3/2}} \\
&\quad - \frac{1}{(p^2L^2 + m^2L^2)^2} + \frac{cosech^2\sqrt{p^2L^2 + m^2L^2}}{2(p^2L^2 + m^2L^2)}]
\end{aligned} \tag{7}$$

For $L \gg m^{-1}$ this gives the usual gaussian limit answer $C_p \sim m^{-(4-D)/2}$, while for $m \to 0$ at finite L, $C_p \sim L^{(4-D)/2}$. The scaling function is provided by I_D.

We now turn to the spherical limit. The propagator $< \phi_{in}(k)\, \phi_{in}(-k) >$ has the structure $(M^2 + k^2 + \frac{n^2\pi^2}{L^2})^{-1}$ where M is the renormalized mass dressed by the bubble [4] in the spherical limit. As usual $M \sim (T - T_c)^{-\nu}$, where in the spherical limit $\nu = (D-2)^{-1}$. The specific heat graphs [4] for the spherical limit are shown in Fig.1. The first graph (1a) corresponds to the gaussian limit. Each additional loop brings an interaction of strength

N^{-1} and a combinatoric factor N (for large N). Thus, each of the graphs (1b,1c,1d....etc) are the same order as the single loop and the sum defines the spherical limit. For $L \to \infty$, the contributions factor and the successive contribution factors are I^2 (1b) ,I^3 (1c)....etc. The total specific heat is proportional to $\frac{I}{1+I}$. In the case of finite geometry this simple geometric series does not obtain as we show below.

To understand the complication, let us look at the two loop graph. The contribution from the graph is ,

$$\begin{aligned}
&= -\frac{1}{L^3}\lambda \sum_{n_1,n_2} \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \frac{d^{D-1}p_2}{(2\pi)^{D-1}} < \phi_{i,n_1}(p_1)\, \phi_{i,n_1}(-p_1) \int dz \sum_{s,t} \\
&\quad \sum_{m_1,m_2,m_3,m_4} \phi_{s,m_1}(k_1)\, \phi_{s,m_2}(k_2)\, \phi_{t,m_3}(k_3)\, \phi_{t,m_4}(-k_1-k_2-k_3) \\
&\quad sin\frac{m_1\pi z}{L}\, sin\frac{m_2\pi z}{L}\, sin\frac{m_3\pi z}{L}\, sin\frac{m_4\pi z}{L} \phi_{j,n_2}(p_2)\, \phi_{j,n_2}(-p_2) > \\
&= -\lambda\frac{N^2}{L^3} \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \frac{d^{D-1}p_2}{(2\pi)^{D-1}} [< \phi_{i,n_1}(p_1)\, \phi_{i,m_1}(-p_1) >]^2 \\
&\quad [< \phi_{j,n_2}(p_2)\phi_{j,m_2}(-p_2) >]^2 \int dz\, sin^2\frac{m_1\pi z}{L}\, sin^2\frac{m_2\pi z}{L} \\
&= -\frac{N}{4L^3} \sum_{n_1,n_2} \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \frac{d^{D-1}p_2}{(2\pi)^{D-1}} [< \phi_{i,n_1}(p_1)\, \phi_{i,n_1}(-p_1) >]^2 \\
&\quad [< \phi_{j,n_2}(p_2)\, \phi_{j,n_2}(-p_2) >]^2 \int dz\, (1+cos\frac{2n_1\pi z}{L})\, (1+cos\frac{2n_2\pi z}{L}) \\
&= -\frac{N}{4L^2} \sum_{n_1,n_2} \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \frac{d^{D-1}p_2}{(2\pi)^{D-1}} [< \phi_{i,n_1}(p_1)\, \phi_{i,n_1}(-p_1) >]^2 \\
&\quad [< \phi_{i,n_2}(p_2)\, \phi_{i,n_2}(-p_2) >]^2 (\, 1 + \frac{1}{2}\delta_{n_1,n_2}\,) \\
&= -\frac{N}{2} [\, I_D^2 + \frac{1}{4L^2} \sum_n \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}} \frac{d^{D-1}p_2}{(2\pi)^{D-1}} \\
&\quad \frac{1}{(p_1^2+M^2+\frac{n^2\pi^2}{L^2})^2} \frac{1}{(p_2^2+M^2+\frac{n^2\pi^2}{L^2})^2} \,] \\
&= -\frac{N}{2} [\, I_D^2 + J_D^{(1)}\,]
\end{aligned} \tag{8}$$

where

$$J_D^{(1)} = \frac{1}{4L^2}\sum_n \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}}\frac{d^{D-1}p_2}{(2\pi)^{D-1}}\frac{1}{(p_1^2+M^2+\frac{n^2\pi^2}{L^2})^2}\frac{1}{(p_2^2+M^2+\frac{n^2\pi^2}{L^2})^2} \quad (9)$$

The difficulty is obvious from Eqn(8). The presence of the term $J_D^{(1)}$ means that the series for the specific heat is not a geometric series. The three loop term analysed in a similar manner yields $\frac{N}{2}(I_D^3 + 2I_D J_D^{(1)} + J_D^{(2)})$, where

$$\begin{aligned} J_D^{(m)} \quad = \quad & \frac{1}{(2L)^{2m}}\sum_n \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}}\frac{d^{D-1}p_2}{(2\pi)^{D-1}}\cdots\frac{d^{D-1}p_{m+1}}{(2\pi)^{D-1}} \\ & \frac{1}{(p_1^2+M^2+\frac{n^2\pi^2}{L^2})^2}\frac{1}{(p_2^2+M^2+\frac{n^2\pi^2}{L^2})^2}\cdots\frac{1}{(p_{m+1}^2+M^2+\frac{n^2\pi^2}{L^2})^2} \end{aligned} \quad (10)$$

The four loop calculation is $-\frac{N}{2}\,[\, I_D^4 \,+\, 3I_D^2 J_D^{(1)} \,+\, 2I_D J_D^{(2)} \,+\, {J_D^{(1)}}^2 \,+\, J_D^{(3)}\,]$. Similarly,the five loop term is $\frac{N}{2}[I_D^5+4I_D^3J_D^{(1)}+3I_D{J_D^{(1)}}^2+2I_DJ_D^{(3)}+2J_D^{(1)}J_D^{(2)}+J_D^{(4)}]$.The contribution from the six loop term is $-\frac{N}{2}[I_D^6+5I_D^4J_D^{(1)}+2I_D^3J_D^{(2)}+6I_D^2{J_D^{(1)}}^2+6I_DJ_D^{(1)}J_D^{(2)}+{J_D^{(1)}}^3+3I_D^2J_D^{(3)}+2I_DJ_D^{(4)}+2J_D^{(1)}J_D^{(3)}+{J_D^{(2)}}^2+J_D^{(5)}]$ and so on.

The pattern is now clear. The specific heat is obtained as

$$\begin{aligned} C \quad &= \quad \frac{N}{2}\,\{\,[I_D - I_D^2 + I_D^3 + ...] - [J_D^{(1)} - J_D^{(2)} + J_D^{(3)} - ..][1 - 2I_D + 3I_D^2 - ..] \\ & \quad -[J_D^{(1)} - J_D^{(2)} + J_D^{(3)} - ..]^2[1 - 3I_D + 6I_D^2 - ..] - [J_D^{(1)} - J_D^{(2)} \\ & \quad +J_D^{(3)} - ..]^3[1 - 4I_D + 10I_D^2 - ..]....\} \\ &= \quad \frac{N}{2}\,\{\,\frac{I_D}{1+I_D} - \frac{\sum_m J_D^{(m)}}{(1+I_D)^2} - \frac{(\sum_m J_D^{(m)})^2}{(1+I_D)^3} - \frac{(\sum_m J_D^{(m)})^3}{(1+I_D)^4} - ..\} \\ &= \quad \frac{N}{2}\,\{\frac{I_D}{1+I_D} - \frac{\sum_m J_D^{(m)}}{(1+I_D)^2}[1 + \frac{\sum_m J_D^{(m)}}{(1+I_D)} + (\frac{\sum_m J_D^{(m)}}{(1+I_D)})^2 + ..\} \\ &= \quad \frac{N}{2}\,\{\frac{I_D}{1+I_D} - \frac{\sum_m J_D^{(m)}}{(1+I_D)^2}\frac{1}{1-\frac{\sum_m J_D^{(m)}}{(1+I_D)}}\,\} \end{aligned}$$

$$= \frac{N}{2}\frac{I_D - \sum_m J_D^{(m)}}{1 + I_D - \sum_m J_D^{(m)}} \tag{11}$$

This is the final answer for any dimension. The first part of Eqn(10), where we generalize the pattern can be proven by induction [12] . We assume that this form is consistent with the form obtained after a m-loop calculation and prove in a long but straightforward fashion that it holds for the $(m+1)$ loop as well. We now write down the explicit expression for I_D and $J_D^{(m)}$ in $D = 3$.

$$I_3 = \frac{1}{2M}\,[\,CothML - \frac{1}{ML}\,] \tag{12}$$

For $L \to \infty$, $I_3 = M^{-1}$ as expected while for $M \to 0$, $I_3 = \frac{L}{3}$ in accordance with finite size scaling approximations. Straightforward algebra yields

$$J_3^{(1)} = -\frac{1}{4L^2}\,\frac{\partial I_3}{\partial M^2} \tag{13}$$

and for general m

$$J_3^{(m)} = \frac{(-)^m}{2^(2m)m!}\,\frac{1}{L^{2m}}\,\frac{\partial^m I_3}{\partial M^{2m}} \tag{14}$$

which allows us to write Eqn(10) as

$$C = \frac{LF(\sqrt{M^2L^2 + \frac{1}{4}})}{1 + LF(\sqrt{M^2L^2 + \frac{1}{4}})} \tag{15}$$

where

$$F(X) = \frac{1}{X}\,[\,CothX - \frac{1}{X}\,] \tag{16}$$

The spherical limit scaling function is contained in Eqns(14) and (15) - the central result of our paper.

We now show how the above equations can be used in the practical situation of liquid He^4 near the superfluid transition in $D = 3$. The experimental specific heat is known to be almost logarithmic, while in the spherical limit

in $D = 3$, $\frac{\alpha}{\nu} = -1$. We can split off a background part from Eqn(14) by writing

$$\begin{aligned} C &= 1 - \frac{1}{1 + LF(\sqrt{M^2L^2 + \frac{1}{4}})} \\ &\simeq 1 - \frac{1}{LF(\sqrt{M^2L^2 + \frac{1}{4}})} \end{aligned} \tag{17}$$

The second step follows from the fact that in the critical region the contribution $LF(\sqrt{M^2L^2 + \frac{1}{4}})$ is expected to be large. It is the factor $\frac{1}{LF(\sqrt{M^2L^2+\frac{1}{4}})}$ which describes the scaling function with $\frac{\alpha}{\nu} = -1$. If we want to write the answer in terms of $\frac{\alpha}{\nu}$, then the specific heat is $[\, LF(\sqrt{X^2 + \frac{1}{4}})\,]^{\frac{\alpha}{\nu}}$ and for $\frac{\alpha}{\nu} \to 0$ as is the experimental situation, we can write

$$\begin{aligned} C(M,L) &= C_0\, ln\, [\, LF(\sqrt{M^2L^2 + \frac{1}{4}})\,] + constant \\ &= C_0\, ln\, [\, \frac{\Lambda L}{\sqrt{M^2L^2 + \frac{1}{4}}}(Coth\sqrt{M^2L^2 + \frac{1}{4}} - \frac{1}{\sqrt{M^2L^2 + \frac{1}{4}}})\,] \end{aligned} \tag{18}$$

where $\Lambda = \kappa_0 t_0^{\nu}$. For $L \to \infty$, the bulk specific heat

$$\begin{aligned} C &= C_0\, ln\, [\, \frac{\Lambda}{M}\,] \\ &= C_0\, ln\, [\, \frac{\Lambda}{\kappa_0 t^{\nu}}\,] \\ &= \nu\, C_0\, ln[\, \frac{t_0}{t}\,] \end{aligned} \tag{19}$$

The experimentally measured bulk value has this form [13] and we identify $\nu C_0 = 5.3$ and $t_0 = \frac{1}{4}$. The value at $M = 0$ is $C_0\, ln\, [2\Lambda L(Coth\frac{1}{2} - 2)]$. Subtracting the $C_0\, ln\, \Lambda L$ from Eqn(18) to obtain a pure scaling part

$$\Delta C(M,L) = C(M,L) - C_0\, ln\, \Lambda L$$

$$= \quad C_0 \, ln \, [\, \frac{(Coth\sqrt{X^2+\frac{1}{4}} - \frac{1}{\sqrt{X^2+\frac{1}{4}}})}{\sqrt{X^2+\frac{1}{4}}} \,] \tag{20}$$

where $X = ML$. We show this plot in Fig(2) and the data of Mehta and Gasparini [14]. The agreement is the proof of the practical utility.

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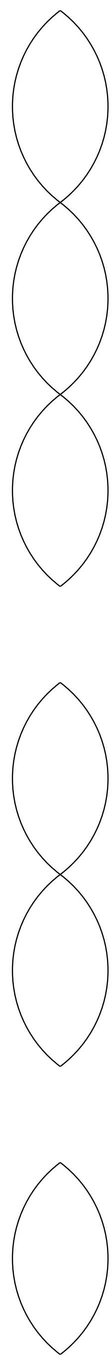
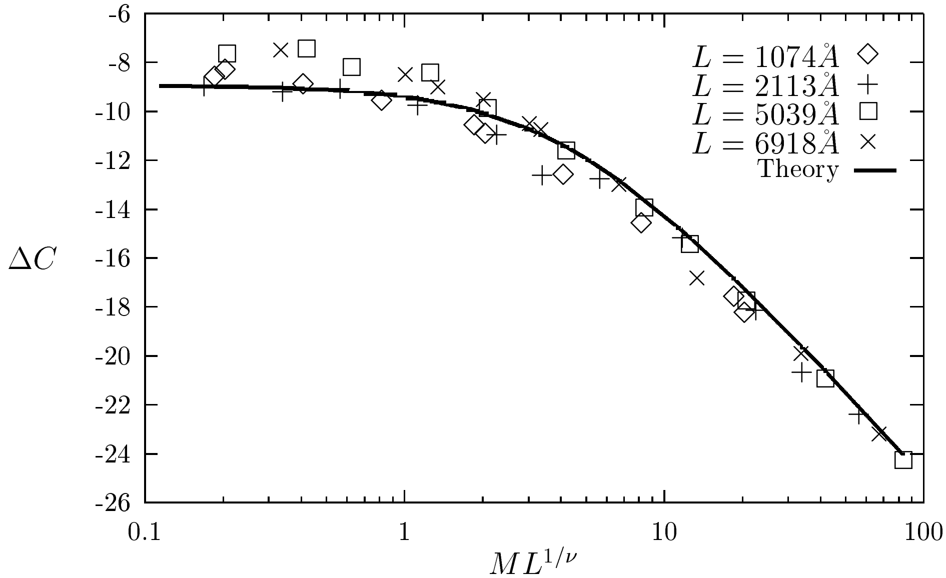
Figure caption

Fig.1. Specific heat graphs in the spherical limit are shown.

Fig.2. $\Delta C = C(M,L) - C_0 \, ln \, \Lambda L$,Eqn(20), plotted against $(ML)^{1/\nu}$. The solid curve refers to our theory and the data is taken from the experiment of Mehta and Gasparini [14]





$L = 1074\mathring{A}$
$L = 2113\mathring{A}$
$L = 5039\mathring{A}$
$L = 6918\mathring{A}$
Theory
ΔC
$ML^{1/\nu}$
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
-26
0.1
1
10
100